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# Recent Problems in Uniform Asymptotic Expansions of Integrals

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Laplace type integrals are considered for large values of the Laplace variable. Additional parameters may have influence on the classical expansion based on Watson's lemma. In that case modifications of the lemma are needed. We construct several uniform expansions in which the extra parameters do not disturb the validity and the nature of the expansions. Applications and examples are discussed for several special functions.

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## 1. Introduction

A well-known lemma in asymptotics is the following

LEMMA (Watson). *Consider the Laplace integral*

$$F(z) = \int_0^\infty e^{-zt} f(t) dt. \tag{1.1}$$

*Assume that*
(i) *$f$ is locally integrable on* $[0, \infty)$;
(ii) $f(t) \sim \sum_{s=0}^{\infty} a_s t^{s+\lambda-1}$ *as* $t \to 0^+$, $\lambda$ *fixed*, $\mathrm{Re}\lambda > 0$;
(iii) *the abscissa of convergence of (1.1) is not* $+\infty$.
*Then*

$$F(z) \sim \sum_{s=0}^{\infty} \Gamma(s+\lambda) a_s z^{-s-\lambda} \tag{1.2}$$

*as* $z \to \infty$ *in the sector* $|\arg z| \leq \frac{1}{2}\pi - \delta (< \frac{1}{2}\pi)$, *where* $z^\lambda$ *has its principal value.*

PROOF. See OLVER (1974, p.113). □

Observe that (1.2) is obtained by substituting (ii) into (1.1) and interchanging the order of summation and integration. In (iii) we assume that (1.1) converges if Re$z$ is sufficiently large (and positive). In (ii) $\lambda$ is fixed. When $\lambda=\mathcal{O}(z)$ (or larger) the expansion (1.2) has no meaning. In that case the ratio of consecutive terms is

$$\Gamma(s+\lambda+1)a_{s+1}z^{-s-1-\lambda}/\Gamma(s+\lambda)a_s z^{-s-\lambda} = \mathcal{O}(\lambda/z), \tag{1.3}$$

if $a_s, a_{s+1} \neq 0$. It follows that the expansion (1.2) looses this asymptotic nature when $\lambda = O(z)$.

In its paper we consider several cases in which Watsons's lemma is not applicable owing to large or small extra parameters in the Laplace integral. These parameters, of which $\lambda$ in the above expansion is a special case, may disturb the given expansion, say (1.2), and their influence can be described in terms of the notion of uniformity. The above expansion is not uniformly valid for $\lambda$ in an unbounded subdomain of Re$\lambda>0$.

We consider the following integrals

$$\int_0^\infty t^{\lambda-1}e^{-zt}f(t)dt \tag{1.4}$$

$$\int_\alpha^\infty t^{\lambda-1}e^{-zt}f(t)dt, \quad \alpha\geqslant 0, \tag{1.5}$$

$$\int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)dt, \quad \alpha\geqslant 0, \tag{1.6}$$

$$\int_0^\infty t^{\lambda-1}e^{-\frac{1}{2}zt^2+\alpha t}f(t)dt, \quad \alpha\in\mathbb{R}, \tag{1.7}$$

and we construct asymptotic expansions with $z$ as large parameter. The parameters $\alpha$ and $\lambda$ play the part of uniformity parameters. For a proper description of the asymptotic estimation of the above integrals we need several special functions as basic approximants, which are obtained by replacing $f$ with a constant, say unity. Then the integrals reduce to:

- gamma function,
- incomplete gamma function,
- Bessel function,
- parabolic cylinder function,

respectively.

## 2. Gamma function as approximant

In fact, the non-uniform expansion (1.2) with $\lambda$ in compact subsets of Re$\lambda>0$ also makes use of gamma functions. The uniform expansion does not need another function. We construct an expansion in which $\lambda$ is allowed to range through the interval $[0,\infty)$. The integral (1.1) is not defined for $\lambda=0$, but instead we use the normalized version

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}f(t)dt. \tag{2.1}$$

Now we have, when $f$ is regular at $t=0$,

$$F_0(z) = f(0), \quad \text{Re}z>0.$$

In the following we assume that $f$ is analytic is a domain $\Omega$ of the complex plane that contains a circle (with positive radius $R$) around the origin and a sector $S_{\alpha,\beta}$ defined by

$$S_{\alpha,\beta} = \{t|-\alpha<\arg t<\beta\} \tag{2.2}$$

where $\alpha,\beta$ are positive numbers. Furthermore, we assume that there is a real number $p$ such that

$$f(t) = \mathcal{O}(t^p),$$

as $t\to\infty$ in $S_{\alpha,\beta}$. The Taylor coefficients of $f$ at $t=0$ are denoted by $a_s$, that is,

$$f(t) = \sum_{s=0}^{\infty} a_s t^s, |t|<R. \tag{2.3}$$

The asymptotic expansion of (2.1) is obtained by "expanding" $f$ around the point $t=\mu$, where $\mu=\lambda/z$. We write

$$F_\lambda(z) = z^{-\lambda}f(\mu) + \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}[f(t)-f(\mu)]dt.$$

Integrating by parts, writing

$$t^\lambda e^{-zt}dt = -\frac{t}{z}\frac{d[e^{-zt}t^\lambda]}{t-\mu},$$

we obtain

$$F_\lambda(z) = z^{-\lambda}f(\mu) + \frac{1}{z\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}f_1(t)dt,$$

where

$$f_1(t) = t\frac{d}{dt}\frac{f(t)-f(\mu)}{t-\mu}.$$

Continuing this procedure, we obtain

$$F_\lambda(z) = z^{-\lambda}[\sum_{s=0}^{n-1} f_s(\mu)z^{-s} + z^{-n}E_n(z,\lambda)], \tag{2.4}$$

where

$$f_{s+1}(t) = t\frac{d}{dt}\frac{f_s(t)-f_s(\mu)}{t-\mu}, \; s = 0,1,..., \tag{2.5}$$

$f_0=f$ and the remainder $E_n$ is given by

$$E_n(z,\lambda) = \frac{z^\lambda}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt}f_n(t)dt. \tag{2.6}$$

From authors papers (1983, 1985) it follows that (2.4) is an asymptotic representation for $z \to \infty$ that holds uniformly with respect to $\mu = \lambda/z$ in a closed sector properly interior to $S_{\alpha,\beta}$ defined in (2.2), and in a disc around the origin. For real values of $\lambda, z$ we can say that

$$E_n(z,\lambda) = O(1) \text{ as } z \to \infty,$$

uniformly with respect to $\lambda$ or $\mu$ in $[0,\infty)$.

This is the generalized version of Watson's lemma that gives expansion (1.2), with values of $f$ (and derivatives) at $t=0$. The new expansion concentrates on values of $f$ at $t=\mu$, the point where $t^{\lambda}e^{-zt}$ attains a maximal value, and the expansion remains valid when $\mu \to 0$ (Watson's lemma) or when $\mu \to \infty$ independent of $z$.

EXAMPLE (Exponential integral) We take $f(t) = 1/(t+1)$. It is easily seen that the function $F_\lambda(z)$ of (1.1) can be written as

$$F_\lambda(z) = z^{1-\lambda} e^{z} E_\lambda(z);$$

$E_\lambda(z)$ is the well-known exponential integral

$$E_\lambda(z) = \int_1^\infty t^{-\lambda} e^{-zt} dt.$$

The first few terms of (2.4) are easily computed and we obtain

$$E_\lambda(z) = \frac{e^{-z}}{z+\lambda}[1 + \frac{\lambda}{(z+\lambda)^2} + \frac{\lambda(\lambda-2z)}{(z+\lambda)^4} + \frac{(z+\lambda)}{z^4} E_3(z,\lambda)]$$

with $E_3$ defined in (2.6) with the functions $f_s$ given in (2.5) with $f_0(t) = 1/(t+1)$. From the first coefficients in this expansion it can be seen that large values of $\lambda$ do not disturb the asymptotic properties of these first coefficients.

When the functions $f_n$ in (2.6) are bounded on $[0,\infty)$, a bound of $|E_n(z,\lambda)|$ can be easily constructed. This gives an error bound for the asymptotic expansion: let positive numbers $M_n$ exists such that, for $n = 0,1,2,\ldots,$

$$|f_n(t)| \leqslant M_n, \quad t \geqslant 0.$$

Then for the remainder in (2.4) we obtain

$$|E_n(z,\lambda)| \leqslant M_n, \quad n = 0,1,2,\ldots.$$

This gives, in a way, an idea of the asymptotic nature of the expansion when $\mu$ is fixed. If the numbers $M_n$ do not depend on $\mu$ (the functions $f_n$ do!) then the expansion holds uniformly with respect to $\mu$. However, it is more realistic to assume that $f_n$ is not bounded on $[0,\infty)$ and/or that $M_n$ depend on $\mu$. This asks for a more detailed approach for constructing error bounds. See author's papers (1985, 1986).

## 3. Incomplete gamma function as approximant

We write (1.5) in the form

$$F_\lambda(z,\alpha) = \frac{1}{\Gamma(\lambda)}\int_\alpha^\infty t^{\lambda-1}e^{-zt}f(t)dt \tag{3.1}$$

and we consider $z$ as the large parameter and $\alpha$ and $\lambda$ as uniformity parameters in $[0,\infty)$; $\alpha=0$ gives the previous case. The saddle point of $t^\lambda e^{-zt}$ at $t=\mu=\lambda/z$ may be inside the domain of integration ($\alpha<\mu$) or outside the domain ($\alpha>\mu$). The transition occurs when $\alpha$ passes the value $\mu$ and this arises interesting asymptotic phenomena, for instance for several types of cumulative distribution functions. For certain combinations of the parameters $\alpha$ and $\mu$ the function $F_\lambda(z,\alpha)$ can be estimated in terms of the normal (i.e. Gaussian) distribution function or error function. For all possible situations in the parameter domain $(\alpha,\mu)\in[0,\infty)x[0,\infty)$ an incomplete gamma function is needed for a uniform expansion.

When $f$ equals unity the function $F_\lambda(z,\alpha)$ reduces to

$$\frac{1}{\Gamma(\lambda)}\int_\alpha^\infty t^{\lambda-1}e^{-zt}dt = z^{-\lambda}Q(\lambda,\alpha z), \tag{3.2}$$

where $Q$ is the incomplete gamma function

$$Q(a,x) = \frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}dt. \tag{3.3}$$

The integration by parts procedure of the previous section now gives an integrated non-vanishing term at $t=\alpha$. So we obtain the formal expansion

$$F_\lambda(z,\alpha)\sim z^{-\lambda}Q(\lambda,\alpha z)\sum_{s=0}^\infty f_s(\mu)z^{-s} + \frac{\alpha^\lambda e^{-\alpha z}}{z\Gamma(\lambda)}\sum_{s=0}^\infty B_s(\alpha)z^{-s}, \tag{3.4}$$

where

$$B_s(\alpha) = \frac{f_s(\alpha)-f_s(\mu)}{\alpha-\mu},\ s = 0,1,..., \tag{3.5}$$

and the functions $f_s$ are the same as in (2.5). Observe that the first series in (3.4) also occurs in (2.4). In fact we can write

$$F_\lambda(\alpha,z) = Q(\lambda,\alpha z)F_\lambda(z) + \frac{\alpha^\lambda e^{-\alpha z}}{z\Gamma(\lambda)}B_\lambda(z,\alpha) \tag{3.6}$$

with $F_\lambda(z)$ defined in (2.1) as the complete integral and

$$B_\lambda(z,\alpha) \sim \sum_{s=0}^\infty \frac{B_s(\alpha)}{z^s}, \tag{3.7}$$

the second series in (3.4). It follows that the present case (3.1) makes use of (2.1) and (2.4) and that, hence, in this section only the function $B_\lambda(z,\alpha)$ matters.

In our (1986) paper we have constructed error bounds for the remainders associated with expansion (3.7). Furthermore, the expansions are applied to the incomplete beta function, which can be transformed into (3.1) by means of a rather complicated transformation.

4. BESSEL FUNCTION AS APPROXIMANT

The integral (1.6) reduces to a modified Bessel function in the case that $f$ is a constant. Explicity we have

$$2(\alpha/z)^{\lambda/2}K_\lambda(2\sqrt{\alpha z}) = \int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}dt. \tag{4.1}$$

In this section we consider

$$F_\lambda(z) = \int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)dt, \tag{4.2}$$

which reduces to the above modified Bessel function in the case that $f$ is a constant.

We construct an asymptotic expansion of the above integral for $z\to\infty$. Observe that for $\alpha=0$ the integral reduces to (2.1); for $\alpha>0$ application of Watson's lemma is not possible due to the essential singularity of $\exp(-\alpha/t)$ at $t=0$.

As a first attempt we may expand $f$ as in Watson's lemma at $t=0$. If we substitute (2.3) into (4.2) we obtain

$$F_\lambda(z) \sim \sum_{s=0}^\infty a_s\Phi_s \tag{4.3}$$

with

$$\Phi_s = 2(\alpha/z)^{(\lambda+s)/2}K_{\lambda+s}(2\sqrt{\alpha z}).$$

Suppose $\alpha$ is a fixed positive number. Then

$$\Phi_{s+1}/\Phi_s = \mathcal{O}(\sqrt{\alpha/z}),\ \text{as}\ z\to\infty. \tag{4.5}$$

On the other hand, if $\alpha z\to 0$, we have

$$\Phi_{s+1}/\Phi_s = \mathcal{O}(z^{-1}). \tag{4.6}$$

This gives an indication that (4.3) may yield an asymptotic expansion of $z\to\infty$, with $\alpha$ restricted to a domain $[0,\alpha_0]$, with $\alpha_0=o(z)$ as $z\to\infty$.

In (4.5), (4.6) we have used the well-known asymptotic estimates

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z},\ \text{as}\ z\to\infty,$$

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(2/z)^\nu,\ \text{as}\ z\to 0.$$

The procedure below gives an expansion that is, under suitable conditions on $f$, uniform with respect to $\alpha\in[0,\infty)$.

We consider (4.2) and we write $\mu^2=\alpha/z$. Saddle points of $\exp(-zt-\alpha/t)$

occur at $t=\pm\mu$, $\mu$ is supposed to be positive. The first step is the representation

$$f(t) = a_0 + b_0 t + (t-\mu^2/t)g(t) \tag{4.7}$$

where $a_0, b_0$ follow from substitution of $t=\pm\mu$. We have

$$a_0 = \tfrac{1}{2}[f(\mu) + f(-\mu)],\ b_0 = \frac{1}{2\mu}[f(\mu)-f(-\mu)].$$

So we obtain upon inserting (4.7) into (4.2)

$$F(z) = a_0\Phi_0 + b_0\Phi_1 + F_1(z) \tag{4.8}$$

where $\Phi_j$ is given in (4.4). An integration by parts gives

$$\begin{aligned} F_1(z) &= \int_0^\infty t^{\lambda-1} e^{-z(t+\mu^2/t)}(t-\mu^2/t)g(t)dt \\ &= -\frac{1}{z}\int_0^\infty t^\lambda g(t) de^{-z(t+\mu^2/t)} \\ &= \frac{1}{z}\int_0^\infty t^{\lambda-1} e^{-z(t+\mu^2/t)} f_1(t)dt, \end{aligned}$$

with $f_1(t)=t^{1-\lambda}\frac{d}{dt}[t^\lambda g(t)]=\lambda g(t)+tg'(t)$. We see that $zF_1(z)$ is of the same form as $F(z)$. The above procedure can now be applied to $zF_1(z)$ and we obtain for (4.2) the formal expansion

$$F(z)\sim\Phi_0 \sum_{s=0}^\infty \frac{a_s}{z^s} + \Phi_1 \sum_{s=0}^\infty \frac{b_s}{z^s},\ \text{as } z\to\infty, \tag{4.9}$$

where we define inductively $f_0(t)=f(t)$, $g_0(t)=g(t)$ and for $s=1,2,...,$

$$f_s(t) = t^{1-\lambda}\frac{d}{dt}[t^\lambda g_{s-1}(t)] = a_s + b_s t + (t-\mu^2/t)g_s(t).$$

$$a_s = \tfrac{1}{2}[f_s(\mu) + f_s(-\mu)],\ b_s = \frac{1}{2\mu}[f_s(\mu)-f_s(-\mu)].$$

By using the recursion relation

$$K_{\nu+1}(z) = K_{\nu-1}(z) + \frac{2\nu}{z}K_\nu(z)$$

the series in (4.3) can be rearranged in the form (4.9); however, then the coefficients are essentially different. In (4.3) $a_s$ comes from $f$ and derivatives of $f$ at $t=0$; in (4.9) $a_s$ and $b_s$ come from function values of $f$ and derivatives of $f$ at $t=\mu$ and $t=-\mu$.

We still need to prove that (4.9) is uniformly valid for $\alpha\in[0,\infty)$. At the moment a proof is not available and the proper conditions on $f$ have to be formulated.

An interesting application of the expansions (4.9) can be given for confluent hypergeometric functions. Let us consider

$$U(a,b,,x) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-xt}dt \tag{4.10}$$

for $a \to +\infty$, with $x>0$, $b \in \mathbb{R}$. A transformation to the standard form (4.2) is needed, but first we give a simple transformation. The function $[t/(1+t)]^a$ takes its maximal value (on $[0,\infty)$) at $t=+\infty$. This function plays the role of an exponential function. Therefore we take as a new variable of integration $\tau$ defined by $t/(1+t)=\exp(-\tau)$. Then (6.9) becomes

$$U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-a\tau - x/(e^\tau - 1)}\tau^{-b}f(\tau)d\tau \tag{4.11}$$

where $f(\tau)=[\tau/(1-e^{-\tau})]^b$. The easiest way to arrive at the standard form (4.2) is to write

$$U(a,b,x) = \frac{e^{\frac{1}{2}x}}{\Gamma(a)}\int_0^\infty \tau^{-b}e^{-a\tau - x/\tau}\tilde{f}(\tau)d\tau \tag{4.12}$$

where

$$\tilde{f}(\tau) = f(\tau)\exp\{x[1/\tau - 1/(e^\tau - 1) - \tfrac{1}{2}]\}.$$

Now we can use the procedure leading to (4.9), with $\lambda=1-b$. The result is an expansion for $a\to\infty$, which holds uniformly with respect to $x\in[0,x_0]$, where $x_0$ is a fixed positive number. It is not possible here to replace $x_0$ by $\infty$; the main reason is that $\tilde{f}(\tau)$ depends on $x$, in such a way that coefficients $a_s$ and $b_s$ in (4.9) grow too fast when $x\to\infty$.

A more powerful expansion is obtained (with respect to the uniformity domain of $x$) when we transform (4.11) into (4.2) by using the mapping $u:\mathbb{R}\to\mathbb{R}$ that is defined by

$$\tau + \frac{\nu}{e^\tau - 1} = u + \frac{\alpha}{u} + A \tag{4.13}$$

where $\nu=x/a$;$\alpha$ and $A$ are to be determined. We compute them by the following condition on the mapping $u$: the critical points at the left-hand side of (4.13) ($\tau=\pm\gamma$, where $\gamma$ is the positive number satisfying $\cosh\gamma=1+\frac{1}{2}\nu$) should correspond with those at the right ($u=\pm\mu$, where $\mu^2=\alpha$). If follows that

$$A = -\tfrac{1}{2}\nu,\ \mu = \tfrac{1}{2}(\gamma+\sinh\gamma). \tag{4.14}$$

These choices make the mapping $u$ regular with $u(0)=0$, $u(\pm\infty)=\pm\infty$. We now obtain from (4.13) and (4.11)

$$U(a,b,x) = \frac{e^{-\frac{1}{2}x}}{\Gamma(a)}\int_0^\infty u^{-b}e^{-a(u+\alpha/u)}f^*(u)du, \tag{4.15}$$

where

$$f^*(u) = (u/\tau)^b f(\tau)\frac{d\tau}{du},$$

with $f$ as in (4.11). We expect that expanding (4.15) as in (4.9) will give $[0,\infty)$ as uniformity domain for $x$. Proofs are needed. The first thing to do is to prove the regularity of $f^*$ in a fixed domain containing $\mathbb{R}$ in its interior. Observe that $f^*$ depends on the uniformity parameter $\alpha$.

After these preparations the asymptotic expansion of the integral in (4.14) can be constructed by computing the coefficients $a_s, b_s$ that appear in (cf. (4.9))

$$e^{\frac{1}{2}x}\Gamma(a)U(a,b,\nu a) \sim \Phi_0 \sum_{s=0}^{\infty}\frac{a_s}{a^s} + \Phi_1 \sum_{s=0}^{\infty}\frac{b_s}{a^s}, \text{ as } a\to\infty, \tag{4.17}$$

where $\Phi_0, \Phi_1$ are given in (4.4) with $\lambda = 1-b$, $z=a$ and $\alpha=\mu^2$, with $\mu$ defined in (4.14). The first coefficients in (4.16) are

$$a_0 = \frac{1}{2}[f^*(\mu) + f^*(-\mu)],\; b_0 = \frac{1}{2\mu}[f^*(\mu) - f^*(-\mu)].$$

A few calculations based on (4.13) and l' Hôpital's rule give

$$\frac{d\tau}{du}\Big|_{u=\pm\mu} = \left[\frac{2(1-e^{-\gamma})}{\mu(1+e^{-\gamma})}\right]^{\frac{1}{2}},\; \mu = \frac{1}{2}[\gamma + \sinh\gamma].$$

To express the first coefficients $a_0, b_0$ in terms of $\gamma$, let $\zeta(\gamma)$ denote the above value of $d\tau/du$ at $u=\pm\mu$, and write

$$\eta(\gamma) = \left[\frac{\mu}{1-e^{-\gamma}}\right]^b.$$

Then we have

$$a_0 = \frac{1}{2}\zeta(\gamma)[\eta(\gamma) + \eta(-\gamma)],\; b_0 = \frac{1}{2\mu}\zeta(\gamma)[\eta(\gamma) - \eta(-\gamma)].$$

Observe that the coefficients contain function values of $f^*$ at the negative axis, although $f^*$ in (4.15) is only used for non-negative $u$-values.

## 5. Parabolic cylinder function as approximant

In the previous section an essential singularity at $t=0$ is incorporated in the Laplace integral. By replacing the exponential function in (2.1) with $\exp(-zt + \alpha\sqrt{t})$ a simpler singularity occurs and in fact this type of singularity can be accepted in Watson's lemma. Then the function $\exp(\alpha\sqrt{t})f(t)$ has to be expanded in a power series. It is more interesting to couple the parameter $\alpha$ with the large parameter $z$ and to consider the effect when $\alpha$ crosses the origin. A slight change of variables gives a quadratic polynomial in the exponential function. In fact we consider

$$I_\lambda(z,\alpha) = \int_0^\infty t^{\lambda-1} e^{-\frac{1}{2}zt^2 + \alpha t} f(t)\,dt \tag{5.1}$$

for a large values of $z$; $\lambda$ is a fixed positive parameter and $\alpha$ a uniformity parameter in $\mathbb{R}$. The saddle point occurs at $t=\alpha/z$. When $\alpha$ is positive it lies inside the interval of integration, when $\alpha$ is negative it is outside the interval. The transition at $\alpha=0$ can be described by using parabolic cylinder functions as approximants, i.e. the above integral with $f(t)\equiv$ constant.

In [1] BLEISTEIN introduced an integration by parts procedure that produced what is now called a canonical expansion. In a way, the procedures of the previous sections are all based on this approach. We repeat the steps in Bleistein's procedure and we also consider a new method for obtaining a similar expansion.

Let $\beta=\alpha/z$ and write

$$f(t) = a_0 + b_0 t + t(t-\beta)g(t), \tag{5.2}$$

with

$$a_0 = f(0),\ b_0 = \frac{f(\beta)-f(0)}{\beta}.$$

Then we have

$$I_\lambda(z,\alpha) = a_0 W_{\lambda-1} + b_0 W_\lambda + J_\lambda(z,\alpha), \tag{5.3}$$

with

$$W_\lambda = \int_0^\infty t^\lambda e^{-\frac{1}{2}zt^2+\alpha t}\,dt, \tag{5.4}$$

a parabolic cylinder function, and

$$J_\lambda(z,\alpha) = \int_0^\infty t^\lambda (t-\beta) e^{-z(\frac{1}{2}t^2-\beta t)} g(t)dt. \tag{5.5}$$

Integrating by parts gives

$$J_\lambda(z,\alpha) = -\frac{1}{z}\int_0^\infty t^\lambda g(t) de^{-z(\frac{1}{2}t^2-\beta t)}$$
$$= \frac{1}{z}\int_0^\infty t^{\lambda-1} e^{-z(\frac{1}{2}t^2-\beta t)} f_1(t)dt$$

with

$$f_1(t) = t^{1-\lambda}\frac{d}{dt}[t^\lambda g(t)].$$

Repeating this process we obtain the above mentioned canonical expansion

$$I_\lambda(z,\alpha) = W_{\lambda-1}\sum_{s=0}^{n-1}\frac{a_s}{z^s} + W_\lambda\sum_{s=0}^{n-1}\frac{b_s}{z^s} + z^{-n}E_n \tag{5.6}$$

with

$$a_s = f_s(0),\ b_s = \frac{f_s(\beta)-f_s(0)}{\beta}$$

$$f_{s+1}(t) = t^{1-\lambda}\frac{d}{dt}[t^{\lambda}\frac{g_s(t)-a_s-b_st}{t(t-\beta)}], \; s=0,1,...,$$

$f_0=f$, and $E_n$ is the remainder given by

$$E_n = \int_0^{\infty} t^{\lambda-1}e^{-z(\frac{1}{2}t^2-\beta t)}f_n(t)dt.$$

Bounds for $|E_n|$ and proofs for the validity of the expansion can be based on bounds for $|f_n(t)|$ on $[0,\infty)$. A complication is that $f_n$ depends also on the uniformity parameter $\beta$.

In a forthcoming paper of Soni & Sleeman the above procedure is replaced with an approach that resembles the procedure in Watson's lemma. Recall that in Watson's lemma (1.2) is obtained by substituting the expansion in (ii). Soni and Sleeman introduce a set of polynomials $\{P_k\}$ satisfying

$$\begin{cases} P_0(t) = 1, \quad P_1(t) = t/(\gamma+1) \\ [t^{\lambda}P_n(t)]' = t^{\lambda}(t-\beta)P_{n-2}(t), \quad n=2,3,.... \end{cases} \tag{5.7}$$

Then they assume that $g$ in (5.2) can be expanded in terms of $\{P_k\}$, writing

$$g(t) = \sum_{k=0}^{\infty} c_k P_k(t), \tag{5.8}$$

where $c_k$ are independent of $t$ and have to be determined. Substituting this expansion in (5.5), we obtain the formal expansion

$$J_{\lambda}(z,\alpha) = \sum_{k=0}^{\infty} c_k\phi_k,$$

$$\phi_k = \int_0^{\infty} t^{\lambda}(t-\beta)e^{-z(\frac{1}{2}t^2-\beta t)}P_k(t)dt.$$

By using the properties of $\{P_k\}$ given in (5.7) it follows that $\phi_k = z^{-1}\phi_{k-2}$ and, hence, that

$$J_{\lambda}(z,\alpha) = \phi_0\sum_{k=0}^{\infty} c_{2k}z^{-k} + \phi_1\sum_{k=0}^{\infty} c_{2k+1}z^{-k},$$

which is of the same form as the expansion in (5.6). There is a simple relation between $c_k$ and $a_k,b_k$. The computation of $c_k$ in (5.8) is not a simpler problem than the computation of $a_s,b_s$ in (5.6). It is expected, however, that this new approach will give new methods for constructing bounds of the remainders in the asymptotic expansion. Soni and Sleeman's method can also be used for other types of integrals.

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